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Dynamics of a stochastic model for continuous flow bioreactor with Contois growth rate

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Abstract In the modelling of the continuous flow bioreactor, due to uncertainties in the environment the growth rate parameter is under perturbation of white noise, which results in a mathematical model governed by a set of stochastic differential equations. In this paper, assume the Contois growth rate is used and then we first show that the stochastic model has always a unique positive solution. Then long time behavior of the model is studied. Our study shows that both the washout and non-washout equilibria are stochastically stable. At the end, we carry out some numerical simulation, which supports our theoretical conclusion well. Also, by the quantities introduced in the last section, both residence time and intensity of the noise have significant effect on the performance of the reactor.

1 Introduction

Wastewater, such as the one from the food industries contains a complex mixture of biodegradable organic materials, such as fresh and partially decomposed food scraps and crop-residues, that may be in suspension or dissolved. The purpose of wastewater treatment is to remove pollutants what can harm the aquatic environment. Let $S(t)$ be the concentration of substrate and $X(t)$ be the concentration of microorganism at time t, respectively. Then after non-dimensionalization, a mathematical model for wastewater treatment process can have the following form.

$$\begin{cases} \frac{dS}{dt} = \frac{1}{\tau}(S_0 - S) - \frac{\mu_m}{\alpha} X g(S, X) - m_S X, \\ \frac{dX}{dt} = \beta \frac{1}{\tau}(X_0 - X) + \gamma \frac{R}{\tau} X + \mu_m X g(S, X) - k_d X, \end{cases} \tag{1.1}$$

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Table 1 Growth rate functions

Type of growth-rate kinetics	Monod's	Contois'	Tessier's
$g =$	$\frac{\mu_m S}{K_s+S}$	$\mu_{\max}\left(\frac{S/X}{K_x+S/X}\right)$	$\mu_m\left(1-\exp\left(-\frac{S}{K_s}\right)\right)$
References	[11,13]	[1,8,11]	[9–12]

Depending on the values of the parameters, Eq. (1.1) represents two types of models: continuous flow reactors used in treatment of industrial wastewater if $\beta = \gamma = 1$ and membrane reactors designed for domestic wastewater treatment if $\gamma = 0$. Function $g(S, X)$ is the growth rate and in the published references three types of growth rate are widely used, which are tabulated as Table 1 although other growth rates, such as Moser, Andrews, Edwards and Luong's, were also used in open literatures [11] and the references therein.

When there is no microorganism in the influent, namely $X_0 = 0$, it is easy to check that the dynamics of a reactor model with idealized recycle is equivalent to the idealized membrane reactor, and that of a reactor model with non-idealized recycle is equivalent to a non-idealized membrane reactor model. In other words, the case of $\beta = \gamma = R = 1$ is equivalent to the case of $\beta = \gamma = 0$ and the case of $0 < \beta < 1, \gamma = 0$ is equivalent to the case of $\beta = \gamma = 1, 0 < R < 1$. Please also see [1,8,9,13] for the cases with specific growth rates.

Reasonably good predications can be provided by deterministic models. However, in reality, uncertainties are always there. Too often these uncertainties are ignored, which limits our prediction. Recently, lots of work in this direction have been carried out and being developed very quickly due to the needs in the related research area [1–7,14], to name but a few.

For the continuous flow reactor models, Chen and Zhang [1] investigated one with Monod's growth rate and uncertainties in the recycle part. They showed that the equilibrium solutions are stochastically stable. Assume due to the noise from the environment, the growth rate parameter, μ_m is perturbed. Then our intension in this paper is to study the effect of this perturbation on the equilibrium solution. If the Contois growth rate is used as in [8], after dimensionalization $g(S, X)$ takes the form of $g(S, X) = \frac{S}{S+X}$. And also the initial concentration, S_0 can be rescaled to 1. More precisely, in this study we have

$$\beta = \gamma = 1, \quad 0 \le R \le 1, \ X_0 = 0, \ S_0 = 1$$

and

$$\mu_m \to \mu_m + \delta dW(t)$$

where σ is the noise intensity. Then we have a stochastic model in Itô form as follows.

$$\begin{cases} \frac{dS}{dt} = \frac{1}{\tau}(1-S) - \frac{\mu_m}{\alpha}\frac{SX}{S+X} - m_S X - \frac{\delta}{\alpha}\frac{SX}{S+X}\frac{dW(t)}{dt}, \\ \frac{dX}{dt} = -\frac{1}{\tau}X + \frac{R}{\tau}X + \mu_m\frac{SX}{S+X} - k_d X + \delta\frac{SX}{S+X}\frac{dW(t)}{dt}. \end{cases} \tag{1.2}$$

When $\sigma = 0$, namely for the deterministic model, the straightforward calculation or reference [8] suggests the model has at most two equilibria: a washout equilibrium $E_1(1, 0)$ and a non-washout equilibrium

$$E_2(S^*, X^*) = \frac{\alpha}{A}\left(1 - R + k_d\tau, \quad R - 1 + (\mu_m - k_d)\tau\right),$$

where

$$A = \alpha(1 - R + k_d\tau) + [(R-1) + (\mu_m - k_d)\tau] * [(m_S\alpha + k_d)\tau + 1 - R].$$

Notice that $0 \le R \le 1$ the non-washout equilibrium E_2 is physically meaningful when

$$\tau > \frac{1-R}{\mu_m - k_d}, \quad 0 < k_d < \mu_m.$$

Then from [8] we know that both E_1 and E_2 are stable when E_2 exists.

The rest of the paper is organised as follows. In Sect. 2, we show that there is a unique nonnegative solution no matter how large the intensities of noise is. Section 3 will demonstrate if the noise satisfied certain condition, then the washout equilibrium E_1 is stochastically asymptotically stable. In Sect. 4, we shall show the stochastic stability of the non-washout equilibrium, E_2. Section 5 concludes the paper with simulations and evaluation of the efficiency of the bioreactor.

2 Existence and uniqueness of the positive solution of model (1.2)

In this section we shall prove that model (1.2) has unique positive solution for given postive initial condition.

Theorem 2.1 *The Eq. (1.2) has a unique solution $(S(t), X(t))$ for $t \in [0, \infty)$ given initial value $(S(0), X(0)) \in R^2_+ = \{x \in R^2 : x_i > 0, i = 1, 2\}$. Furthermore, the solution will remain in R^2_+ with probability 1, namely $(S(t), X(t)) \in R^2_+$ for all $t \ge 0$ almost surely.*

Proof Since the coefficients of model (1.2) are locally lipschitz continuous for any given value $(S(0), X(0)) \in R^2_+$, there is a unique local solution $(S(t), X(t))$ for t on $[0, \tau_e)$, where τ_e is the explosion time. Next we prove that this solution is global by showing that $\tau_e = \infty$ *a.s.*. For this purpose, let $m_0 > 0$ be sufficiently large so that $S(0) \in [\frac{1}{m_0}, m_0]$, $X(0) \in [\frac{1}{m_0}, m_0]$. For each integer $m \ge m_0$, define the stopping time as the following

$$\tau_m = \inf\left\{t \in [0, \tau_e) : S(t) \in \left(\frac{1}{m}, m\right) \text{ or } X(t) \in \left(\frac{1}{m}, m\right)\right\},$$

where throughout this paper, we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_m is increasing as $m \to \infty$ and $\tau_\infty = \lim_{m\to\infty} \tau_m \le \tau_e$ *a.s.*. Then in the rest of this section, we only need to show that $\tau_\infty = \infty$ *a.s.*.

If this statement is not true, then for any given $T > 0$ there is a $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon.$$

Hence there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \varepsilon \tag{2.1}$$

for all $m \geq m_1$. Define a C^2-function V: $R_+^2 \to \overline{R}_+$,

$$V(S(t), X(t)) = c(S - 1 - \ln S) + (X - 1 - \ln X),$$

where c is a positive constant to be determined. Then $V(S, X) \geq 0$ since $u - 1 - \ln u \geq 0, \forall u > 0$. Using Itô's formula, we get

$$\begin{aligned} dV &= cdS - \frac{c}{S}dS + \frac{c}{2S^2}(dS)^2 + dX - \frac{1}{X}dX + \frac{1}{2X^2}(dX)^2 \\ &= LVdt + \left(\frac{\alpha\delta - c\delta}{\alpha}\frac{SX}{S+X} + \frac{c\delta}{\alpha}\frac{X}{S+X} - \delta\frac{S}{S+X}\right)dW(t), \end{aligned}$$

where

$$\begin{aligned} LV = {} & \frac{c}{\tau}(1-S) - \frac{c}{\alpha}\mu_m\frac{SX}{S+X} - cm_S X - \frac{c}{S\tau}(1-S) + \frac{c}{\alpha}\mu_m\frac{X}{S+X} \\ & + cm_S\frac{X}{S} + \frac{c}{2S^2}\frac{\delta^2}{\alpha^2}\frac{S^2X^2}{(S+X)^2} - \frac{1}{\tau}X + \frac{R}{\tau}X + \mu_m\frac{SX}{S+X} - k_d X + \frac{1}{\tau} - \frac{R}{\tau} \\ & - \mu_m\frac{S}{S+X} + k_d + \frac{\delta^2}{2X^2}\frac{S^2X^2}{(S+X)^2}. \end{aligned}$$

Note that $R \in [0, 1]$ is the recycle rate, and $S(t)$ and $X(t)$ are real numbers satisfied $0 < S(t) < \infty,\ 0 < X(t) < \infty$. We choose $c = \alpha$, then

$$\begin{aligned} LV = {} & \frac{\alpha}{\tau}(1-S) - \mu_m\frac{SX}{S+X} - \alpha m_S X - \frac{\alpha}{S\tau} + \frac{\alpha}{\tau} + \mu_m\frac{X}{S+X} + \alpha m_S\frac{X}{S} \\ & + \frac{\delta^2}{2\alpha}\frac{X^2}{(S+X)^2} - \frac{1}{\tau}X + \frac{R}{\tau}X + \mu_m\frac{SX}{S+X} - k_d X + \frac{1}{\tau} - \frac{R}{\tau} - \mu_m\frac{S}{S+X} \\ & + k_d + \frac{\delta^2}{2}\frac{S^2}{(S+X)^2} \leq 2\frac{\alpha}{\tau} + \mu_m + \alpha m_S K_1 + \frac{\delta^2}{2\alpha} + \frac{1}{\tau} - \frac{R}{\tau} + k_d + \frac{\delta^2}{2} \triangleq K. \end{aligned}$$

Therefore,

$$\begin{aligned} & \int_0^{\tau_m \wedge T} dV(S(r), X(r)) \\ & \quad \leq \int_0^{\tau_m \wedge T} K dr + \int_0^{\tau_m \wedge T} \left(\frac{\alpha\delta - c\delta}{\alpha}\frac{SX}{S+X} + \frac{c\delta}{\alpha}\frac{X}{S+X} - \delta\frac{S}{S+X}\right)dW(t). \end{aligned}$$

Taking expectation on each side of the above inequality yields

$$\begin{aligned} E[V(S(\tau_m \wedge T), X(\tau_m \wedge T))] &\le V(S(0), X(0)) + E\int_0^{\tau_m \wedge T} K dr \\ &\le V(S(0), X(0)) + KT. \end{aligned} \tag{2.2}$$

Let $\Omega_m = \{\tau_m \le T\}$ for $m \ge m_1$. From (2.1), we have $P(\Omega_m) \ge \varepsilon$. Note that for every $\omega \in \Omega_m$, there is at least one of $S(\tau_m, \omega)$, $X(\tau_m, \omega)$ equals either m or $\frac{1}{m}$. If $S(\tau_m, \omega) = m$ or $\frac{1}{m}$, then

$$V(S(\tau_m \wedge T), X(\tau_m \wedge T)) \ge c(m - 1 - c \ln m) \wedge c\left(\frac{1}{m} - 1 - \ln \frac{1}{m}\right),$$

while if $X(\tau_m, \omega) = m$ or $\frac{1}{m}$, then

$$V(S(\tau_m \wedge T), X(\tau_m \wedge T)) \ge (m - 1 - \ln m) \wedge \left(\frac{1}{m} - 1 - \ln \frac{1}{m}\right).$$

Consequently,

$$\begin{aligned} &V(S(\tau_m \wedge T), X(\tau_m \wedge T)) \\ &\quad \ge c(m-1-\ln m) \wedge c\left(\frac{1}{m} - 1 - \ln \frac{1}{m}\right) \wedge (m - 1 - \ln m) \wedge \left(\frac{1}{m} - 1 - \ln \frac{1}{m}\right). \end{aligned}$$

It then follows from Eqs. (2.1) and (2.2) that

$$\begin{aligned} &V(S(0), X(0)) + KT \\ &\quad \ge E\left[1_{\Omega_m(\omega)} V(S(\tau_m \wedge T), X(\tau_m \wedge T))\right] \\ &\quad \ge c(m-1-\ln m) \wedge c\left(\frac{1}{m} - 1 - \ln \frac{1}{m}\right) \wedge (m - 1 - \ln m) \wedge \left(\frac{1}{m} - 1 - \ln \frac{1}{m}\right). \end{aligned}$$

where $1_{\Omega_m(\omega)}$ is the indicator function of Ω_m. Letting $m \to \infty$ leads to the contradiction $\infty > V(S(0), X(0)) + KT = \infty$. It implies that $\tau_\infty = \infty$ $a.s.$. This completes the proof. □

3 Stochastically asymptotical stability of the washout equilibrium of (1.2)

From Sect. 1 we know for the deterministic model, the washout equilibrium $E_1(1, 0)$ is always stable when $R - 1 < (k_d - \mu_m)\tau$. And it is easy to verify that it is still the equilibrium of the stochastic model (1.2). In this section, for the stochastic case we shall investigate the stability of E_1. In other words, we are interested in the effect of uncertainty on the stability of E_1. The main result of this section can be stated as follows.

Theorem 3.1 *The washout equilibrium $E_1(1,0)$ of model (1.2) is stochastically asymptotically stable when the intensity of the noise, δ satisfies*

$$\delta^2 < 2\left(\frac{1-R}{\tau} + k_d - \mu_m\right)\frac{\alpha^2}{1+\alpha^2}.$$

Proof Firstly, we shift the equilibrium solution of (1.2) to the origin by letting $X_1(t) = S(t) - 1,\ X_2(t) = X(t)$, which results a model of the following form

$$\begin{cases} dX_1 = (-\frac{1}{\tau}X_1 - \frac{\mu_m}{\alpha}\frac{(X_1+1)X_2}{1+X_1+X_2} - m_s X_2)dt - \frac{\delta}{\alpha}\frac{(X_1+1)X_2}{1+X_1+X_2}dW(t), \\ dX_2 = (-\frac{1}{\tau}X_2 + \frac{R}{\tau}X_2 + \mu_m\frac{(X_1+1)X_2}{1+X_1+X_2} - k_d X_2)dt + \delta\frac{(X_1+1)X_2}{1+X_1+X_2}dW(t). \end{cases} \tag{3.1}$$

Secondly, we linearize (3.1) at the origin, (0, 0) of the new coordinate system. The linearization is given by

$$\begin{cases} dX_1 = \left(-\frac{1}{\tau}X_1 - \frac{\mu_m}{\alpha}X_2 - m_s X_2\right)dt - \frac{\delta}{\alpha}X_2 dW(t), \\ dX_2 = \left(-\frac{1}{\tau}X_2 + \frac{R}{\tau}X_2 + \mu_m X_2 - k_d X_2\right)dt + \delta X_2 dW(t). \end{cases} \tag{3.2}$$

Our proof will then be splitted into 2 steps: (1) To prove the stability of the equilibrium of the linearized model (3.2) and (2) To prove the stability of the equlibrium of the nonlinear model (3.1). According this idea, next we show the equilibrium $(X_1, X_2) = (0, 0)$ of model (3.2) is stochastically asymptotically stable. To this end, define a C^2-function V as

$$V = X_1^2 + X_2^2 + AX_2,$$

where A is a positive constant. Obviously, V is positive definite, and along the trajectories of model (3.2) we have

$$\begin{aligned} dV &= 2X_1 dX_1 + (dX_1)^2 + 2X_2 dX_2 + (dX_2)^2 + AdX_2 \\ &= LVdt + \left(-2\frac{\delta}{\alpha}X_1X_2 + 2\delta X_2^2 + A\delta X_2\right)dW(t), \end{aligned}$$

where

$$\begin{aligned} LV = &-\frac{2}{\tau}X_1^2 - \frac{2\mu_m}{\alpha}X_1X_2 - 2m_s X_1X_2 + \frac{\delta^2}{\alpha^2}X_2^2 - \frac{2}{\tau}X_2^2 + \frac{2R}{\tau}X_2^2 + 2\mu_m X_2^2 \\ &-2k_d X_2^2 + \delta^2 X_2^2 + A\left(-\frac{1}{\tau} + \frac{R}{\tau} + \mu_m - k_d\right)X_2. \end{aligned}$$

We claim that LV is a negative function. In fact,

(i) If $X_1 \geq 0$, then

$$LV \leq -\frac{2}{\tau}X_1^2 + \left(\frac{2R}{\tau} - \frac{2}{\tau} + \frac{\alpha^2+1}{\alpha^2}\delta^2 + 2\mu_m - 2k_d\right)X_2^2.$$

Obviously when $\delta^2 < 2(\frac{1-R}{\tau} + k_d - \mu_m)\frac{\alpha^2}{1+\alpha^2}$, $LV \leq 0$, and $LV = 0$ if and only if $X_1 = X_2 = 0$.

(ii) If $X_1 < 0$, we can get $-X_1 < 1$ from $S = X_1 + 1 > 0$, then

$$LV \leq -\frac{2}{\tau}X_1^2 + \frac{2\mu_m}{\alpha}X_2 + 2m_s X_2 + \frac{\delta^2}{\alpha^2}X_2^2 - \frac{2}{\tau}X_2^2 + \frac{2R}{\tau}X_2^2 + 2\mu_m X_2^2 \\ -2k_d X_2^2 + \delta^2 X_2^2 + A\left(-\frac{1}{\tau} + \frac{R}{\tau} + \mu_m - k_d\right)X_2.$$

We then can choose

$$A = \frac{\frac{2\mu_m}{\alpha} + 2m_s}{\frac{1}{\tau} - \frac{R}{\tau} - \mu_m + k_d},$$

which yields

$$LV \leq -\frac{2}{\tau}X_1^2 + \left(\frac{2R}{\tau} - \frac{2}{\tau} + \frac{\alpha^2+1}{\alpha^2}\delta^2 + 2\mu_m - 2k_d\right)X_2^2.$$

Again, when $\delta^2 < 2\left(\frac{1-R}{\tau} + k_d - \mu_m\right)\frac{\alpha^2}{1+\alpha^2}$, $LV \leq 0$, and $LV = 0$ if and only if $X_1 = X_2 = 0$. Therefore the (0, 0) origin model (3.2) is globally stochastic asymptotically stable.

Next, we show the origin (0,0) of model (3.1) is also stochastically asymptotically stable, namely the washout equilibrium $E_1(1, 0)$ of model (1.2) is stochastically asymptotically stable. Notice

$$\begin{aligned}
&|f(t,X) - F\cdot X| + |g(t,X) - G\cdot X| \\
&= \sqrt{\left(\frac{\mu_m}{\alpha}X_2 - \frac{\mu_m}{\alpha}\frac{(X_1+1)X_2}{1+X_1+X_2}\right)^2 + \left(\mu_m\frac{(X_1+1)X_2}{1+X_1+X_2} - \mu_m X_2\right)^2} \\
&\quad + \sqrt{\left(\frac{\delta}{\alpha}X_2 - \frac{\delta}{\alpha}\frac{(X_1+1)X_2}{1+X_1+X_2}\right)^2 + \left(\delta\frac{(X_1+1)X_2}{1+X_1+X_2} - \delta X_2\right)^2} \\
&= \sqrt{2\left(\frac{S_0}{1+S_0}X_2 - \frac{(X_1+S_0)X_2}{1+X_1+S_0}\right)^2} \\
&= \sqrt{\left(\frac{\mu_m^2}{\alpha^2}+\alpha^2\right)\left[\frac{(X_1+1)X_2}{1+X_1+X_2} - X_2\right]^2} + \sqrt{\left(\frac{\delta^2}{\alpha^2}+\alpha^2\right)\left[\frac{(X_1+1)X_2}{1+X_1+X_2} - X_2\right]^2} \\
&= \left(\sqrt{\frac{\mu_m^2}{\alpha^2}+\alpha^2} + \sqrt{\frac{\delta^2}{\alpha^2}+\alpha^2}\right)\left|\frac{(X_1+1)X_2}{1+X_1+X_2} - X_2\right| \\
&= \left(\sqrt{\frac{\mu_m^2}{\alpha^2}+\alpha^2} + \sqrt{\frac{\delta^2}{\alpha^2}+\alpha^2}\right)\left|\frac{X_2^2}{1+X_1+X_2}\right|.
\end{aligned}$$

Then by [5] and [7], for small $\varepsilon > 0$, when $|X_1| < \varepsilon$, $|X_2| < \varepsilon$, choose $l = 2\max\left\{\sqrt{\frac{\mu_m^2}{\alpha^2}+\alpha^2}, \sqrt{\frac{\delta^2}{\alpha^2}+\alpha^2}\right\}$,

$$\begin{aligned}&|f(t,X)-F\cdot X|+|g(t,X)-G\cdot X|\\ &\quad\le l\varepsilon\left|\frac{X_2}{1+X_1+X_2}\right|\\ &\quad\le l\varepsilon.\end{aligned}$$

Therefore the washout equilibrium $E_1(1,0)$ of model (1.2) is stochastically asymptotically stable. This completes the proof. □

4 Asymptotic behavior of the non-washout equilibrium of the deterministic model

Generally, E_2 is not an equilibrium of stochastic model (1.2) any more when $\sigma \neq 0$. However, since model (1.2) can be treated as the perturbation of model (1.1) which has an non-washout equilibrium E_2, it is reasonable to consider the microorganism will be persist if the solution of model (1.2) is going around E_2 at the most of time. In this sense, we have conclusion as follows.

Theorem 4.1 *Let* $(S(t), X(t))$ *be the solution of model* (1.2) *with initial value* $(S(0), X(0)) \in R_+^2$. *Then when* $\delta^2 \le \min\left\{\frac{1}{4\tau}, \left[\frac{2-2R}{\tau}+2k_d+2\alpha m_S+\left(\frac{2\alpha-\alpha R}{\tau}+\alpha k_d+\alpha^2 m_S\right)\left(\frac{2S^*}{X^*}-\frac{1}{X^*}\right)\right]*\frac{1}{8}\right\}$,

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left[\left(S(u)-S^*\right)^2+r^2\left(X(u)-X^*\right)^2\right]du \le k_\sigma,$$

where

$$\begin{aligned}r^2 &= \frac{8\delta^2-\frac{2}{\tau}+\frac{2R}{\tau}-2k_d-2\alpha m_S-\left(\frac{2\alpha-\alpha R}{\tau}+\alpha k_d+\alpha^2 m_S\right)\left(\frac{2S^*}{X^*}-\frac{1}{X^*}\right)}{8\delta^2\alpha^2-\frac{2\alpha^2}{\tau}},\\ k_\sigma &= \frac{c}{\frac{2\alpha^2}{\tau}-8\delta^2\alpha^2},\\ c &= 8\delta^2\left(-RX^*+\tau k_d X^*+\alpha\tau m_S X^*\right)^2\\ &\quad+4\delta^2\alpha^2+\left(\frac{2\alpha-\alpha R}{\tau}+\alpha k_d+\alpha^2 m_S\right)\left(\frac{1}{X^*}+\frac{(S^*+X^*)\,\delta}{\mu_m}\right).\end{aligned}$$

Proof Define a C^2-function V as follows

$$V = [\alpha(S - S^*) + (X - X^*)]^2 + \left(\frac{4\alpha - 2\alpha R}{\tau} + 2\alpha k_d + 2\alpha^2 m_S\right) \times \frac{(S^* + X^*)}{\mu_m X^*}\left(X - X^* - X^* \ln \frac{X}{X^*}\right),$$

where $\delta^2 \leq \min\left\{\frac{1}{4\tau}, \left[\frac{2-2R}{\tau} + 2k_d + 2\alpha m_S + \left(\frac{2\alpha - \alpha R}{\tau} + \alpha k_d + \alpha^2 m_S\right)\left(\frac{2S^*}{X^*} - \frac{1}{X^*}\right)\right] * \frac{1}{8}\right\}$. Then V is positive definite. Using Itô's formula, we get

$$\begin{aligned} dV &= 2\left[\alpha(S - S^*) + (X - X^*)\right](\alpha dS + dX) + \alpha^2(dS)^2 + (dX)^2 \\ &\quad + \left(\frac{4\alpha - 2\alpha R}{\tau} + 2\alpha k_d + 2\alpha^2 m_S\right)\frac{(S^* + X^*)}{\mu_m X^*}\left[\left(1 - \frac{X^*}{X}\right)dX + \frac{X^*}{2X^2}(dX)^2\right] \\ &= LVdt + \left(\frac{4\alpha - 2\alpha R}{\tau} + 2\alpha k_d + 2\alpha^2 m_S\right)\frac{(S^* + X^*)}{\mu_m X^*}\left(X - X^*\right)\delta\frac{S}{S + X}dW(t), \end{aligned}$$

where

$$\begin{aligned} LV &= 2\left[\alpha(S - S^*) + (X - X^*)\right]\left(\frac{\alpha}{\tau} - \frac{\alpha}{\tau}S - \alpha m_S X - \frac{1}{\tau}X + \frac{R}{\tau}X - k_d X\right) \\ &\quad + 2\delta^2\frac{S^2X^2}{(S + X)^2} + \left(\frac{4\alpha - 2\alpha R}{\tau} + 2\alpha k_d + 2\alpha^2 m_S\right)\frac{(S^* + X^*)}{\mu_m X^*} \\ &\quad \times \left[(X - X^*)\mu_m\left(\frac{S}{S + X} - \frac{S^*}{S^* + X^*}\right) + \frac{X^*\delta S^2}{2(S + X)^2}\right]. \end{aligned}$$

Note that $-\frac{1}{\tau}X^* + \frac{R}{\tau}X^* + \mu_m\frac{S^*X^*}{S^*+X^*} - k_d X^* = 0$ implies $\mu_m\frac{S^*}{S^*+X^*} = \frac{1}{\tau} - \frac{R}{\tau} + k_d$. Then we have

$$\frac{\alpha}{\tau} = \frac{\alpha}{\tau}S^* + \frac{1}{\tau}X^* - \frac{R}{\tau}X^* + k_d X^* + \alpha m_S X^*$$

and

$$\begin{aligned} &[2\alpha(S - S^*) + 2(X - X^*)]\left(\frac{\alpha}{\tau} - \frac{\alpha}{\tau}S - \alpha m_S X - \frac{1}{\tau}X + \frac{R}{\tau}X - k_d X\right) \\ &= [2\alpha(S - S^*) + 2(X - X^*)]\left[-\frac{\alpha}{\tau}(S - S^*) + \left(-\frac{1}{\tau} + \frac{R}{\tau} - k_d - \alpha m_S\right)\left(X - X^*\right)\right] \\ &= \left(-\frac{2\alpha^2}{\tau}(S - S^*\right)^2 + \left(-\frac{2}{\tau} + \frac{2R}{\tau} - 2k_d - 2\alpha m_S\right)\left(X - X^*\right)^2 \\ &\quad + \left(-\frac{4\alpha}{\tau} + \frac{2\alpha R}{\tau} - 2\alpha k_d - 2\alpha^2 m_S\right)\left(X - X^*\right)(S - S^*), \end{aligned}$$

which imply the following hold.

$$\begin{aligned}
2\delta^2\frac{S^2X^2}{(S+X)^2} &\le 2\delta^2X^2\\
&\le 4\delta^2[(X+\alpha S-\alpha)^2+\alpha^2]\\
&= 4\delta^2[(X+\alpha S-\alpha S^*-X^*+RX^*-\tau k_dX^*-\alpha\tau m_SX^*)^2+\alpha^2]\\
&= 4\delta^2[(X-X^*)+\alpha(S-S^*)+(RX^*-\tau k_dX^*-\alpha\tau m_SX^*)]^2\\
&\quad+4\delta^2\alpha^2\le 8\delta^2(X-X^*)^2+8\alpha^2\delta^2(S-S^*)^2+8\delta^2(-RX^*\\
&\quad+\tau k_dX^*+\alpha\tau m_SX^*)^2+4\delta^2\alpha^2,
\end{aligned}$$

and note that $X^*\in[0,1]$ and $S^*\in[0,1]$, $S(t)$ and $X(t)$ are real numbers satisfied $0<S(t)<\infty$, $0<X(t)<\infty$, then

$$\begin{aligned}
&\left(\frac{4\alpha-2\alpha R}{\tau}+2\alpha k_d+2\alpha^2m_S\right)\frac{(S^*+X^*)}{\mu_mX^*}\\
&\quad\times\left[(X-X^*)\mu_m\left(\frac{S}{S+X}-\frac{S^*}{S^*+X^*}+\frac{X^*\delta S^2}{2(S+X)^2}\right)\right]\\
&=\left(\frac{4\alpha-2\alpha R}{\tau}+2\alpha k_d+2\alpha^2m_S\right)\frac{(S^*+X^*)}{\mu_mX^*}\\
&\quad\times\left[(X-X^*)\mu_m\frac{SX^*-S^*X}{(S+X)(S^*+X^*)}+\frac{X^*\delta S^2}{2(S+X)^2}\right]\\
&\le\left(\frac{4\alpha-2\alpha R}{\tau}+2\alpha k_d+2\alpha^2m_S\right)\frac{1}{X^*}(X-X^*)\frac{SX^*-S^*X+1}{S+X+1}\\
&\quad+\left(\frac{2\alpha-\alpha R}{\tau}+\alpha k_d+\alpha^2m_S\right)\frac{(S^*+X^*)\,\delta}{\mu_m}\\
&\le\left(\frac{4\alpha-2\alpha R}{\tau}+2\alpha k_d+2\alpha^2m_S\right)\frac{1}{X^*}(X-X^*)\left(SX^*-S^*X^*+S^*X^*-S^*X+1\right)\\
&\quad+\left(\frac{2\alpha-\alpha R}{\tau}+\alpha k_d+\alpha^2m_S\right)\frac{(S^*+X^*)\,\delta}{\mu_m}\\
&\le\left(\frac{4\alpha-2\alpha R}{\tau}+2\alpha k_d+2\alpha^2m_S\right)\left(X-X^*\right)\left(S-S^*\right)\\
&\quad-\left(\frac{4\alpha-2\alpha R}{\tau}+2\alpha k_d+2\alpha^2m_S\right)\frac{S^*}{X^*}(X-X^*)^2\\
&\quad+\left(\frac{4\alpha-2\alpha R}{\tau}+2\alpha k_d+2\alpha^2m_S\right)\frac{1}{X^*}(X-X^*)\\
&\quad+\left(\frac{2\alpha-\alpha R}{\tau}+\alpha k_d+\alpha^2m_S\right)\frac{(S^*+X^*)\,\delta}{\mu_m}\\
&\le\left(\frac{4\alpha-2\alpha R}{\tau}+2\alpha k_d+2\alpha^2m_S\right)(X-X^*)(S-S^*)\\
&\quad-\left(\frac{4\alpha-2\alpha R}{\tau}+2\alpha k_d+2\alpha^2m_S\right)\frac{S^*}{X^*}\left(X-X^*\right)^2
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{4\alpha-2\alpha R}{\tau}+2\alpha k_d+2\alpha^2 m_S\right)\frac{1}{X^*}\left[\frac{(X-X^*)^2+1}{2}\right]\\
&+\left(\frac{2\alpha-\alpha R}{\tau}+\alpha k_d+\alpha^2 m_S\right)\frac{(S^*+X^*)\,\delta}{\mu_m}\\
=&\left(\frac{4\alpha-2\alpha R}{\tau}+2\alpha k_d+2\alpha^2 m_S\right)(X-X^*)(S-S^*)\\
&-\left[\left(\frac{4\alpha-2\alpha R}{\tau}+2\alpha k_d+2\alpha^2 m_S\right)\frac{S^*}{X^*}\right.\\
&\left.-\left(\frac{2\alpha-\alpha R}{\tau}+\alpha k_d+\alpha^2 m_S\right)\frac{1}{X^*}\right](X-X^*)^2\\
&+\left(\frac{2\alpha-\alpha R}{\tau}+\alpha k_d+\alpha^2 m_S\right)\frac{(S^*+X^*)\delta}{\mu_m}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
LV \le &\left(8\alpha^2\delta^2-\frac{2\alpha^2}{\tau}\right)\left(S-S^*\right)^2\\
&+\left[8\delta^2-\frac{2}{\tau}+\frac{2R}{\tau}-2k_d-2\alpha m_S\right.\\
&\left.-\left(\frac{2\alpha-\alpha R}{\tau}+\alpha k_d+\alpha^2 m_S\right)\left(\frac{2S^*}{X^*}-\frac{1}{X^*}\right)\right]\left(X-X^*\right)^2+c.
\end{aligned}
$$

where

$$
\begin{aligned}
c=&\,8\delta^2\left(-RX^*+\tau k_d X^*+\alpha\tau m_S X^*\right)^2+4\delta^2\alpha^2\\
&+\left(\frac{2\alpha-\alpha R}{\tau}+\alpha k_d+\alpha^2 m_S\right)\left(\frac{1}{X^*}+\frac{(S^*+X^*)\,\delta}{\mu_m}\right).
\end{aligned}
$$

From

$$
E\int_0^t dV=E\int_0^t LV dt,
$$

we know

$$
\lim_{t\to\infty}\sup\frac{1}{t}E\int_0^t\left[(S(u)-S^*)^2+r^2\left(X(u)-X^*\right)^2\right]du\le k_\sigma,
$$

where $r^2$and k_σ are defined in the theorem statement. This completes the proof. □

5 Numerical simulation and conclusion

In this section, we first carry out numerical simulations to demonstrate the stochastic stability of the equilibrium solutions, E_1 and E_2, and then use the pre-defined functions

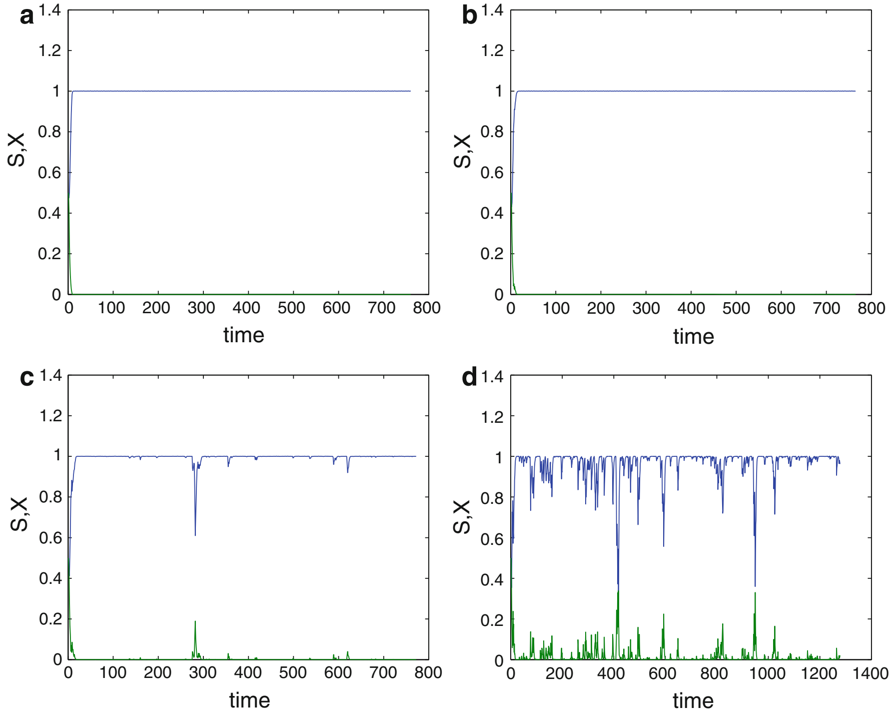

Fig. 1 Comparison of the dynamics in deterministic model and stochastic model with $\delta = 0.2, 0.3, 0.5$, respectively

to assess the performance of the bioreactor modeled in this paper. Our simulations agree well with our theoretical analysis in previous Sects. 3 and 4. The performance of the bioreactor can be affected by the noise in certain degree.

5.1 Stochastic stability of E_1 and E_2

First, we demonstrate washout equilibrium E_1 is stochastically stable. To this end, we choose $(S(0), X(0)) = (0.5, 0.5)$ as the initial value in model (1.2), and let $R = 0.8, \alpha = 1, k_d = 0.6, \mu_m = 0.4, m_S = 0.5, \tau = 2$, and $\delta = 0.1$. Since $R - 1 = -0.2$, $(k_d - \mu_m)\tau = 0.4$ which satisfies $R - 1 < (k_d - \mu_m)\tau$ and $\delta^2 < 2(\frac{1-R}{\tau} + k_d - \mu_m)\frac{\alpha^2}{1+\alpha^2} = 0.3$, from Sect. 3, the equilibrium $E_1(1, 0)$ is globally asymptotically stable. Our simulation also supports this conclusion as shown in Fig. 1, where we show the effect from different intensities, δ of noises. As seen, when δ^2 is relatively small such as less than 0.1, there are not much effect from the noise; when δ^2 is large, but less than 0.3 it is still stable although the effect is obvious. In practice, it implies if the uncertainty can be controlled within certain range, the microorganism will die out eventually.

Next we show E_2 is stochastically stable too. For this purpose, in our simulation, the parameters are set as $R = 0.8, \alpha = 1, k_d = 0.4, \mu_m = 0.6, m_S = 0.5, \tau = 2$.

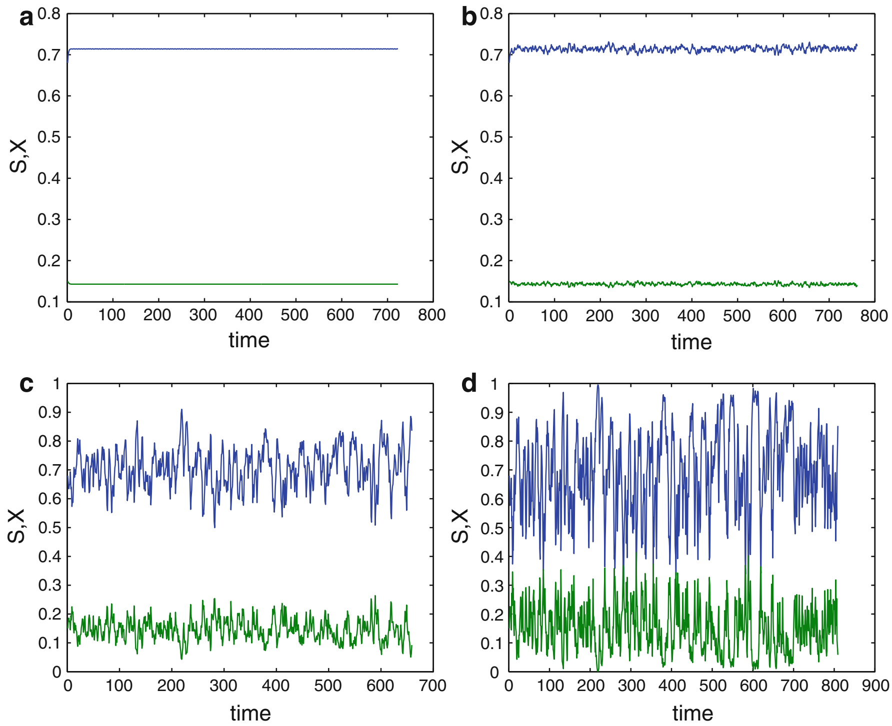

Fig. 2 Comparison of the dynamics in deterministic model and stochastic model with $\sigma = 0.06, 0.1, 0.3$, respectively

Since $\tau > \frac{1-R}{\mu_m - k_d} = 1, 0 < k_d < \mu_m$, model (1.1) has an unique positive equilibrium $E_2(S^*, X^*) = (\frac{5}{7}, \frac{1}{7})$, which according to the result in Sect. 4, when $\sigma^2 \le \min\{\frac{\tau}{2}, \tau - \frac{\tau R}{2} + \frac{k_d \tau}{2} + \frac{m_s \tau}{2}\} = \min\{1, 1.9\} = 0.125$, is stable. Our simulation in this case uses $(0.66, 0.18)$ as the initial value. It shows in Fig. 2 that the solution of the model (1.2) goes around the point $E_2(S^*, X^*)$ at the most of time, which implies E_2 is stochastically stable when σ satisfies the condition in Theorem 4.1.

While for $\delta^2 \le \min\{\frac{1}{4\tau}, [\frac{2-2R}{\tau} + 2k_d + 2\alpha m_S + (\frac{2\alpha - \alpha R}{\tau} + \alpha k_d + \alpha^2 m_S)(\frac{2S^*}{X^*} - \frac{1}{X^*})] * \frac{1}{8}\} = \min\{0.125, 0.625\} = 0.125$ E_2 is unstable, please see Fig. 3, which shows the long time behavior of the model (1.2) with $\delta = 0.8, 1.8$ respectively.

The implication of Figs. 2 and 3 is that in practice the microorganisms can persist if the noise is controlled in certain level due to the positive equilibrium is stochastically stable.

5.2 Performance of the bioreactor

In order to quantitatively assess the performance of the bioreactor, we introduce the following dimensionless quantities, for more details please see [8] and the references therein. Let (S^*, X^*) be the equilibrium solution as defined in the previous sections.

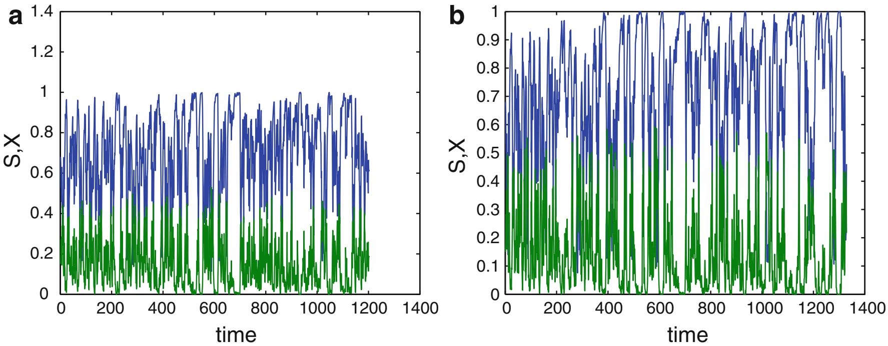

Fig. 3 The long time behavior of the stochastic model with $\delta = 0.8, 1.8$, respectively

Then we have the specific utilisation and yield defined by

$$\mathcal{U} = \frac{S_0 - S^*}{X^*}\frac{1}{\tau} \text{ and } \mathcal{Y} = \frac{X^*}{S_0 - S^*} \tag{5.1}$$

respectively. In what follows we introduce the treatment/process efficiency

$$\mathcal{E} = 100 \times \frac{S_0 - S^*}{S_0}, \tag{5.2}$$

the rate of waste treatment

$$\mathcal{W} = \frac{S_0 - S^*}{\tau} \tag{5.3}$$

and the effective yield

$$\mathcal{Y}_e = \frac{X^*}{S_0}. \tag{5.4}$$

Notice that from (1.1)

$$\frac{S_0 - S^*}{X^*} = \frac{(m_S\alpha + k_d)\tau + \beta - \gamma R}{\alpha} \tag{5.5}$$

which implies that the utilization and yield are independent of the noise intensity, σ. Please also see the first subfigure of Fig. 4. It illustrates the yield, $\mathcal{Y}$ against the residence time for different values of σ. However, for the effective yield ($\mathcal{Y}_e$), rate of waste treatment ($\mathcal{W}$) and treatment efficiency ($\mathcal{E}$) they are decreasing when σ increases from zero at first and then increasing as it gets bigger, which implies there is certain value for $\sigma > 0$ which corresponds a minimum performance of the reactor, please see the subfigures 2–4 of Fig. 4. Also Fig. 4 indicates that both the utilisation and the yield are decreasing functions of the residence time, τ; treatment efficiency is a increasing

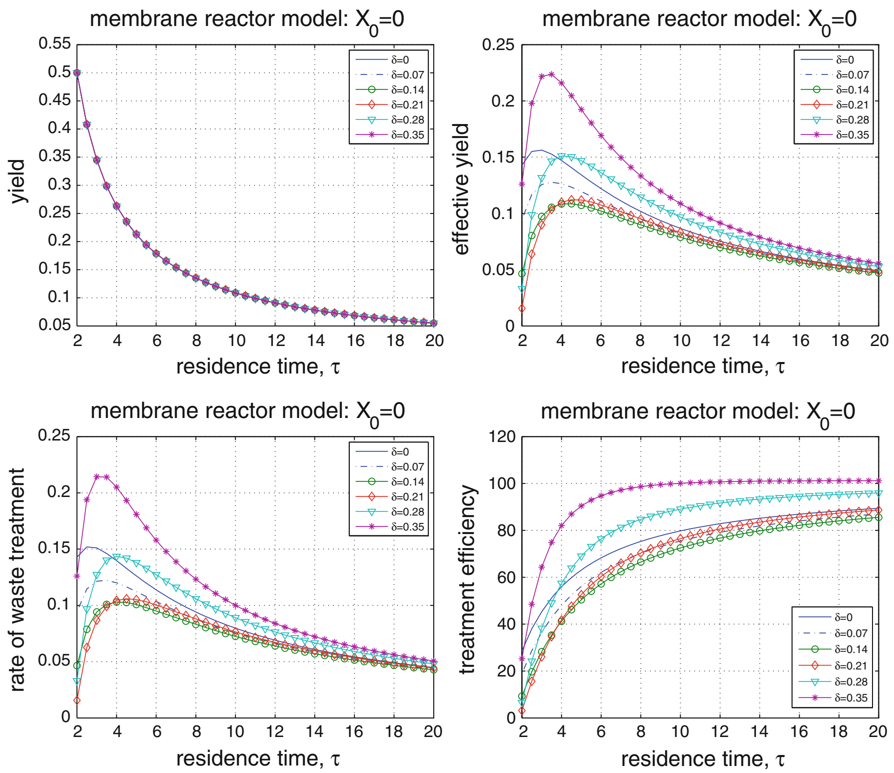

Fig. 4 Performance of the bioreactor

function of the residence time, τ; and both effective yield and rate of waste treatment are increasing as the residence time grows from small value, then reach their maxima and then decrease as τ getting larger. In other word, larger residence time results in a higher treatment efficiency; however lower utilisation, yield, effective yield and rate of wasttreatment.

To sum up, in this paper, we first proposed a stochastic model and then analyzed the long time dynamics of a stochastic model. Then proved the existence of the positive solution. The conditions for globally stochastic stability of the washout and non-washout equilibria have been given. At the end, numerical simulations have also been carried out, which support our theoretical analysis in previous sections.

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